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MAM1020S **Tutorial 2/3** **August 2017**

Note that for the exams the following rule will apply so better getting use to it already with the tutorials: no credit will be given to unjustified answers. Justify all your answers completely. (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could loose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will loose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...
Questions in MAGENTA and RED are harder. In any test it will be much easier.

1. (a) Consider the set of points $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $1 = (x-1)^2 + (y-1)^2$. Is this set of points define the graph of a function?

   Solution: Take $x = 1$, then $(1, 2)$ and $(1, 0)$ are both in the previous set of points. But, if it was a function then to one x would correspond a unique y by definition, thus this cannot be a function.

   (b) Consider the set of points $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $y = x^2$. Is this set of points define the graph of a function?

   Solution: Yes this is the graph of a function as it is defined to associate to each $x \in \mathbb{R}$ a unique $y = x^2 \in \mathbb{R}$.

2. Find the domain of each of the following functions: that is all the real values for which the following function are defined:

   (a) $f(x) = \dfrac{4 + \sqrt{2 - 3x}}{4 - \sqrt{2 + x^2}}$

   Solutions: f is defined as a quotient of two functions $x \mapsto 4 + \sqrt{2 - 3x}$ and $x \mapsto 4 - \sqrt{2 + x^2}$. Thus it is only defined when the denominator of $f(x)$ is non zero, that is $4 - \sqrt{2 + x^2} \neq 0$. Moreover, we can see that some square roots are involved in the definition of f and square roots are only defined over $\mathbb{R}^+$. Thus, in order for f to be defined we also need that $2 - 3x \geq 0$ and $2 + x^2 \geq 0$. For any other value of of x, f will be defined as a algebraic combination of function defined. In summary, we need to find the $x \in \mathbb{R}$ such that $4 - \sqrt{2 + x^2} \neq 0$, $2 + x^2 \geq 0$ and

$2-3x \geq 0$. Note that $4-\sqrt{2+x^2}=0$ if and only if $\sqrt{2+x^2}=4$ that is $2+x^2=16 \Leftrightarrow x^2=14 \Leftrightarrow x=\pm\sqrt{14}$. Note also that x^2+2 is always positive and $2-3x \geq 0 \Leftrightarrow x \leq 2/3$. As a conclusion, f is define on $(-\infty, 2/3]\backslash\{-\sqrt{14}\}$.

(b) $g(x) = 3\sqrt{\sin x - 1} + 2\sqrt{-x^3+1}$

Solution: The function g is a linear combination of the function $g_1 : \mathbb{R} \to \mathbb{R}$ defined by $g_1(x) = \sqrt{\sin x - 1}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ defined by $g_2(x) = \sqrt{-x^3+1}$. Thus the domain of definition is the intersection of the domain of definition of g_1 and the one of g_2. g_1 is a square root function, thus only defined for the value of such that $sin(x)-1 \geq 0$. Similarly, g_2 is a square root function thus only defined for the value of x such that $-x^3+1 \geq 0$. Note that $sin(x)-1 \geq 0 \Leftrightarrow sin(x) \geq 1$. Since, we have that $-1 \leq sin(x) \leq 1$, then the only possibility for this is that $sin(x) = 1$ and that is the case when $x \in \{\pi/2 + 2\pi k : k \in \mathbb{Z}\}$. Now $-x^3+1 \geq 0 \Leftrightarrow 1 \geq x^3 \Leftrightarrow x \leq 1$ as we are taking the $\sqrt[3]{.}$ the cube root which is a increasing function over $\mathbb{R}$. The domain of definition of g is the intersection between $(-\infty, 0]$ and $\{\pi/2+2\pi k : k \in \mathbb{Z}\}$. Note that $\pi/2 + 2\pi k \leq 1 \Leftrightarrow k \leq \frac{1-\pi/2}{2\pi} < 0$. Thus, the domain of definition of g is $\{\pi/2 + 2\pi k : k \text{ is a negative integer}\}$

(c) $h(x) = \dfrac{x-5}{x^3-3x^2-x}$

Solutions: h is a rational function, thus defined when its denominator is non-zero. That is for each real value x such that $x^3-3x^2-x \neq 0$. Note that $x^3-3x^2-x = 0 \Leftrightarrow x(x^2-3x-1) = 0 \Leftrightarrow x = 0$ *or* $x^2-3x-1 = 0$. Moreover x^2-3x-1 is a quadratic polynomial whose discriminant is $(-3)^2 - 4 \times 1 \times (-1) = 9+4 = 13$. Thus its roots are $\frac{3\pm\sqrt{13}}{2}$. As a consequence, h is defined over $\mathbb{R} - \{0, \frac{3\pm\sqrt{13}}{2}\}$.

(d) $f(x) = \sqrt{2x^2-3x+4}$

Solutions: f is a square root function, thus defined for the values of x such that $2x^2-3x+4 \geq 0$. $2x^2-3x+4$ is a quadratic polynomial whose discriminant is $(-3)^2 - 4 \times 2 \times 4 = -7$ is negative. Thus, this polynomial has the same sign as 2 over $\mathbb{R}$. That is, $2x^2-3x+4 \geq 0$, for any $x \in \mathbb{R}$. As a consequence, the domain of definition of f is all $\mathbb{R}$.

3. We define the piecewise function $f : [0, +\infty) \to \mathbb{R}$

$$f(x) = \begin{cases} x, & if\ 0 \leq x \leq 1 \\ 2 - x, & if\ 1 < x \leq 2 \\ 0, & if\ x > 2 \end{cases}$$

(a) What is the domain of f? codomain of f?

Solutions: The domain of f is $[0, +\infty)$. The codomain of f is $\mathbb{R}$.

(b) Compute $f(0)$, $f(1/2)$, $f(1)$, $f(2)$, $f(3)$.

Solutions: Since $0, 1/2, 1 \in [0, 1]$, $f(0) = 0$, $f(1/2) = 1/2$, $f(1) = 1$. Since $2 \in]1, 2]$, $f(2) = 2 - 2 = 0$ and since $3 > 2$, $f(3) = 0$.

(c) Compute the image of 5 by f.

Solutions: The image of 5 by f is $f(5)$ and $5 > 2$, thus $f(5) = 0$.

(d) Is the point $(1, 2)$ in $Graph(f)$?

Solution: Since $1 \in [0, 1]$, $f(1) = 1 \neq 2$. Thus, $(1, 2)$ is not in $Graph(f)$.

(e) Draw the graph. (Explain how you obtained it.)

Solution: The graph of f goes from $[0, +\infty)$. Since over $[0, 1]$ the function f is given by $f(x) = x$. It is a line. In order to get a line we need two points since $f(0) = 0$ and $f(1) = 1$ thus this segment passes through $(0, 0)$ and $(1, 1)$. Since over $(1, 2]$, the function is given by $f(x) = 2 - x$ thus it is again a segment and since $f(1) = 2 - 1 = 1$ and $f(2) = 2 - 2 = 0$, thus this segment passes through $(1, 1)$ and $(2, 0)$. Finally the function is given by $f(x) = 0$ over $(2, +\infty)$, thus the graph for this function is the x-axis.

(f) Find all the preimages, if they exist, for $1, 0$ and 5

Solution: The preimages of 1 are the x such that $f(x) = 1$. Since f is a piece wise we need to split the study through the different intervals where it changes definition. Over $[0, 1]$, we have that f is defined by $f(x) = x$ and thus the only preimage of 1, is $x = 1$. Over $(1, 2]$, we have $f(x) = 2 - x$ and $2 - x = 1 \Leftrightarrow x = 1$ again. Finally over $(2, +\infty)$, we have $f(x) = 0$ thus $f(x)$ will never be 1. Thus, 1 has only one preimage over $[0, +\infty)$ which is 1.

The preimages of 0 are the x such that $f(x) = 0$. Since f is a piece wise we need to split the study through the different intervals where it changes definition. Over $[0, 1]$, we have that f is defined by $f(x) = x$ and thus the only preimage of 0, is $x = 0$. Over $(1, 2]$, we have $f(x) = 2 - x$ and $2 - x = 0 \Leftrightarrow x = 2$. Finally over $(2, +\infty)$, we have $f(x) = 0$ thus $f(x)$ will never be 0 all the time, and every element in $(2, +\infty)$ are preimage. Thus, $[2, +\infty) \cup \{0\}$ is the set of all the preimages of 0 over $[0, +\infty)$.

The preimages of 5 are the x such that $f(x) = 5$. Since f is a piece wise we need to split the study through the different intervals where it changes definition. Over $[0, 1]$, we have that f is defined by $f(x) = x$ and thus there is no preimage of 5 when x varies through $[0, 1]$. Over $(1, 2]$, we have $f(x) = 2 - x$ and $2 - x = 5 \Leftrightarrow x = -3$ again no preimage here belonging to $(1, 2]$. Finally over $(2, +\infty)$, we have $f(x) = 0$ thus $f(x)$ will never be 5. Thus, 5 has no preimage over $[0, +\infty)$.

(g) Is f one-to-one over $[0, +\infty[$? onto $\mathbb{R}$? bijective?

Solution: We just proven that there are infinitely many preimages for 0 thus 0 has more than one preimage proving that f is not one-to-one. Also we have proven that 5 has no preimage thus f is not onto. As a consequence f is not bijective since it is not one-to-one for instance.

(h) Compute the range of f.

Solutions: The range of f is given by

$$\begin{aligned} Range(f) &= \{f(x) : x \in [0, +\infty)\} \\ &= \{f(x) : x \in [0, 1]\} \cup \{f(x) : x \in [1, 2]\} \cup \{f(x) : x \in (2, +\infty)\} \\ &= \{x : x \in [0, 1]\} \cup \{2 - x : x \in [1, 2]\} \cup \{0 : x \in (2, +\infty)\} \end{aligned}$$

Clearly $\{x : x \in [0, 1]\} = [0, 1]$ and $\{0 : x \in (2, +\infty)\} = \{0\}$. Moreover,

$$1 \leq x \leq 2 \Leftrightarrow -2 \leq -x \leq -1 \Leftrightarrow 0 \leq 2 - x \leq 1$$

then $\{2 - x : x \in [1, 2]\} = [0, 1]$.

Thus $Range(f) = [0, 1]$.

(i) Compute $f([0, 1/2])$ and $f(\{1, 2\})$.

Solution:

$$f([0, 1/2]) = \{f(x) : x \in [0, 1/2]\} = \{x : x \in [0, 1/2]\} = [0, 1/2]$$

and

$$f(\{1, 2\}) = \{f(1), f(2)\} = \{1, 2 - 2\} = \{1, 0\}$$

(j) Compute $f^{-1}([0, 1])$ and $f^{-1}(\{-2, -1, 0\})$.

Solution:

$$\begin{array}{rcl} f^{-1}([0,1]) & = & \{x \in [0,+\infty) : f(x) \in [0,1]\} \\ & = & \{x \in [0,1], f(x) \in [0,1]\} \\ & & \cup\{x \in [1,2], f(x) \in [0,1]\} \\ & & \cup\{x \in (2,+\infty), f(x) \in [0,1]\} \\ & = & \{x \in [0,1], x \in [0,1]\} \\ & & \cup\{x \in [1,2], 2-x \in [0,1]\} \\ & & \cup\{x \in (2,+\infty), 0 \in [0,1]\} \end{array}$$

Since $\{x \in [0,1], x \in [0,1]\} = [0.1]$, $1 \leq x \leq 2 \Leftrightarrow -2 \leq -x \leq -1 \Leftrightarrow 0 \leq 2-x \leq 1$, so $\{x \in [1,2], 2-x \in [0,1]\} = [0,1]$ and $\{x \in (2,+\infty), 0 \in [0,1]\} = \{0\}$. Thus $f^{-1}([0,1]) = [0,1]$.

(k) Give the intervals where f is increasing? decreasing? constant?

Solution: f is clearly increasing over $[0,1]$, f is decreasing over $[1,2]$ since for any $x_1, x_2 \in [1,2]$, if $x_1 \leq x_2$, then $2-x_1 \geq 2-x_2$, thus $f(x_1) \geq f(x_2)$. Finally, f is constant equal to 0 over $(2,+\infty)$.

4. Consider the functions

$$f: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \to & -3x^2+4x+1 \end{array},$$

$$g: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \to & -3x^2+7/3 \end{array}$$

and

$$h: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \to & x-2/3 \end{array}$$

(a) Prove that for any $x \in \mathbb{R}$, $f(x) = -3(x-2/3)^2+7/3$.

Solution: for any $x \in \mathbb{R}$,

$$-3(x-2/3)^2+7/3 = -3(x^2-4/3x+4/9)+7/3 = -3x^2+4x+1 = f(x)$$

(b) Using the previous question find the roots f, that is all the values $x \in \mathbb{R}$ such that $f(x) = 0$.

Solution:

$$\begin{array}{rcl} f(x) & = & 0 \Leftrightarrow -3(x-2/3)^2+7/3 = 0 \\ & \Leftrightarrow & (x-2/3)^2-7/9 = 0 \\ & \Leftrightarrow & (x-2/3-\sqrt{7}/3)(x-2/3+\sqrt{7}/3) = 0 \end{array}$$

Thus either $x = 2/3+\sqrt{7}/3$ or $x = 2/3-\sqrt{7}/3$. As a conclusion, the preimages of 0 are $\{2/3+\sqrt{7}/3, 2/3-\sqrt{7}/3\}$.

(c) Compute the all the preimages of 5, -2.

Solutions: The preimages of 5 are the x such that $f(x) = 5$, that is

$$\begin{aligned} f(x) &= 5 \Leftrightarrow -3(x-2/3)^2 + 7/3 = 5 \\ &\Leftrightarrow (x-2/3)^2 - 7/9 = -5/3 \\ &\Leftrightarrow (x-2/3)^2 + 8/9 = 0 \\ &\Leftrightarrow (x-2/3)^2 = -8/9 \end{aligned}$$

That has no solutions. Thus there is no preimages of 5.

The preimages of -2 are the x such that $f(x) = -2$, that is

$$\begin{aligned} f(x) &= -2 \Leftrightarrow -3(x-2/3)^2 + 7/3 = -2 \\ &\Leftrightarrow (x-2/3)^2 - 7/9 = 2/3 \\ &\Leftrightarrow (x-2/3-1/3)(x-2/3+1/3) = 0 \\ &\Leftrightarrow (x-1)(x-1/3) = 0 \end{aligned}$$

Thus the preimages of -1 are $\{1, 1/3\}$.

(d) Compute $f(t^2)$, $f(s-1)$, $f(-x_1)$ and $f(x+2/3)$.

Solutions:

$$f(t^2) = -3(t^2)^2 + 4t^2 + 1 = -3t^4 + 4t^2 + 1,$$

$$f(s-1) = -3(s-1)^2 + 4(s-1) + 1 = -3(s^2-2s+1) + 4s - 4 + 1 = -3s^2 + 10s - 6$$

$$f(-x_1) = -3(-x_1)^2 + 4(-x_1) + 1 = -3x_1^2 - 4x_1 + 1$$

$$\begin{aligned} f(x+2/3) &= -3(x+2/3)^2 + 4(x+2/3) + 1 \\ &= -3(x^2 + 4/3x + 4/9) + 4x + 8/3 + 1 \\ &= -3x^2 + 7/3 \end{aligned}$$

(e) Give two points in $Graph(f)$.

Solution: Since $f(0) = -3 \times 0^2 + 4 \times 0 + 1 = 1$ then $(0, 1) \in Graph(f)$ and $f(1) = -3 \times 1^2 + 4 \times 1 + 1 = 2$, thus $(1, 2) \in Graph(f)$.

(f) Prove that for any $x \in \mathbb{R}$, $f(x) = g \circ h(x)$. Remember $g \circ h(x) = g(h(x))$.

Solution: For any $x \in \mathbb{R}$,

$$g \circ h(x) = g(h(x)) = g(x - 2/3) = -3(x-2/3)^2 + 7/3 = f(x)$$

see question 1. Thus the result.

(g) Prove that h is increasing over $\mathbb{R}$.

Solution: Since for any $x_1, x_2 \in \mathbb{R}$, if $x_1 \leq x_2$, then $x_1 - 2/3 \leq x_2 - 2/3$. That is, $h(x_1) \leq h(x_2)$. Thus h is increasing.

(h) Prove that g is not increasing over $\mathbb{R}$. But that g is increasing over $(-\infty, 0]$ and decreasing over $[0, +\infty)$.

Solution: Take $x_1 = 0$ and $x_2 = 1$, we have $x_1 \leq x_2$, also $g(x_1) = g(0) = -3 \times 0 + 7/3 = 7/3$ and $g(x_2) = g(2) = -3 \times 1^2 + 7/3 = -2/3$ thus $g(x_1) > g(x_2)$. Proving that g is not increasing.
When $x_1, x_2 \in (-\infty, 0]$, if $x_1 \leq x_2$, then $x_1^2 \geq x_2^2$, since the square function is decreasing over $(-\infty, 0]$ and $-3x_1^2 \leq -3x_2^2$, then $-3x_1 + 7/3 \leq -3x_2 + 7/3$. Finally, $g(x_1) \leq g(x_2)$. Thus g is is increasing $(-\infty, 0]$.
When $x_1, x_2 \in [0, +\infty)$, if $x_1 \leq x_2$, then $x_1^2 \leq x_2^2$, since the square function is increasing over $[0, +\infty)$ and $-3x_1^2 \geq -3x_2^2$, then $-3x_1 + 7/3 \geq -3x_2 + 7/3$. Finally, $g(x_1) \geq g(x_2)$. Thus g is is decreasing $(-\infty, 0]$.

(i) Deduce that f is increasing over $(-\infty, 2/3]$ and decreasing over $[2/3, +\infty)$.

Solution: Let $x_1, x_2 \in (-\infty, 2/3]$, $x_1 \leq x_2$ thus $x_1 - 2/3 \leq x_2 - 2/3 \leq 0$, since g is increasing over $(-\infty, 0]$, then

$$g(x_1 - 2/3) \leq g(x_2 - 2/3),$$

Thus $f(x_1) \leq f(x_2)$. Thus f is increasing over $(-\infty, 2/3]$.
Let $x_1, x_2 \in [2/3, +\infty)$, $x_1 \leq x_2$ thus $0 \leq x_1 - 2/3 \leq x_2 - 2/3$, since g is decreasing over $[0. + \infty)$, then

$$g(x_1 - 2/3) \geq g(x_2 - 2/3),$$

Thus $f(x_1) \geq f(x_2)$. Thus, f is decreasing $[2/3, +\infty)$.

(j) Compute $f([0, 1])$, $f(\{1, 2, 3\})$.

Solution:

$$f([0,1]) = \{f(x) : x \in [0,1]\} = \{f(x) : x \in [0, 2/3]\} \cup \{f(x) : x \in [2/3, 1]\}$$

When $0 \leq x \leq 2/3$, $f(0) \leq f(x) \leq f(2/3)$ since f is increasing over $[0, 2/3]$, $1 \leq f(x) \leq 7/3$. Moreover, when $2/3 \leq x \leq 1$, f is decreasing, thus $f(1) \leq f(x) \leq f(2/3)$. Moreover $f(1) = 2$ and $f(2/3) = 7/3$, thus $2 \leq f(x) \leq 7/3$. Then,

$$\{f(x) : x \in [0, 2/3]\} = [1, 7/3] \; and \; \{f(x) : x \in [2/3, 1]\} = [2, 7/3]$$

and

$$f([0, 1]) = [1, 7/3].$$

We have $f(1) = 2$, $f(2) = -3 \times 2^2 + 4 \times 2 + 1 = -12 + 8 + 1 = -3$ and $f(3) = -3 \times 3^2 + 4 \times 3 + 1 = -27 + 12 + 1 = -14$. Then,

$$f(\{1, 2, 3\}) = \{-14, -3, 2\}$$

(k) Compute $f^{-1}([0,1])$, $f^{-1}(\{1,2,3\})$.

Solution:

$$\begin{aligned} f^{-1}([0,1]) &= \{x \in \mathbb{R} : f(x) \in [0,1]\} \\ &= \{x \in \mathbb{R} : -3(x-2/3)^2 + 7/3 \in [0,1]\} \end{aligned}$$

Note that

$$0 \leq -3(x-2/3)^2 + 7/3 \leq 1 \Leftrightarrow 7/9 \geq (x-2/3)^2 \geq 4/9$$

Remark: If $a \geq 0$, $x^2 \leq a \Leftrightarrow x^2 - a = (x-\sqrt{a})(x+\sqrt{a}) \leq 0$ By doing a sign table you get that $x^2 \leq a \Leftrightarrow -\sqrt{a} \leq x \leq \sqrt{a}$. Moreover, $x^2 \geq a \Leftrightarrow x^2 - a = (x-\sqrt{a})(x+\sqrt{a}) \geq 0$ By doing a sign table you get that $x^2 \geq a \Leftrightarrow x \leq -\sqrt{a}$ and $x \geq \sqrt{a}$.

Since $7/9 \geq (x-2/3)^2 \Leftrightarrow -\frac{\sqrt{7}}{3} \leq x - 2/3 \leq \frac{\sqrt{7}}{3} \Leftrightarrow 2/3 - \frac{\sqrt{7}}{3} \leq x \leq \frac{\sqrt{7}}{3} + 2/3$.

Moreover,

$$(x-2/3)^2 \geq 4/9 \Leftrightarrow x \geq 2/3 + 2/3 = 4/3 \text{ or } x \leq -2/3 + 2/3 = 0$$

Thus

$$\begin{aligned} f^{-1}([0,1]) &= [2/3 - \tfrac{\sqrt{7}}{3}, \tfrac{\sqrt{7}}{3} + 2/3] \cup (-\infty, 0] \cup [4/3, +\infty) \\ &= [2/3 - \tfrac{\sqrt{7}}{3}, 0] \cup [4/3, \tfrac{\sqrt{7}}{3} + 2/3] \end{aligned}$$

$$\begin{aligned} f^{-1}(\{1,2,3\}) &= \{x \in \mathbb{R} : f(x) \in \{1,2,3\}\} \\ &= \{x \in \mathbb{R} : f(x) = 1 \text{ or } f(x) = 2 \text{ or } f(x) = 3\} \end{aligned}$$

Note that

$$f(x) = 1 \Leftrightarrow -3(x-2/3)^2 + 7/3 = 1 \Leftrightarrow -3(x-2/3)^2 = -4/3 \Leftrightarrow (x-2/3)^2 = 4/9$$

That is equivalent to $x - 2/3 = 2/3$ or $x - 2/3 = -2/3$. That is, $x = 4/3$ or $x = 0$.

$$f(x) = 2 \Leftrightarrow -3(x-2/3)^2 + 7/3 = 2 \Leftrightarrow -3(x-2/3)^2 = -1/3 \Leftrightarrow (x-2/3)^2 = 1/9$$

That is equivalent to $x - 2/3 = 1/3$ or $x - 2/3 = -1/3$. That is, $x = 1$ or $x = 1/3$.

$$f(x) = 3 \Leftrightarrow -3(x-2/3)^2 + 7/3 = 3 \Leftrightarrow -3(x-2/3)^2 = 2/3 \Leftrightarrow (x-2/3)^2 = -2/9$$

That is impossible since a square is always positive, thus this equation has no solution.

As a conclusion,

$$f^{-1}(\{1,2,3\}) = \{0, 1/3, 4/3, 1\}$$

(l) Is f one-to-one? onto? bijective?

Solution : We just seen that $f(4/3) = f(0) = 1$ thus we found two different point $4/3$ and 0 having the same image 1. Thus f is not onto. We have also proven in the previous question that 3 has no preimage at all, thus f is not onto. Finally f is not bijective since f is not one-to-one for instance.

(m) Prove that g is even.

Solution: Let $x \in \mathbb{R}$,

$$g(-x) = -3(-x)^2 + 7/3 = -3x^2 + 7/3 = g(x)$$

Thus g is even.

(n) Prove that f is neither odd nor even.

Solution: Let's us consider $x = 1$,

$$f(-1) = -3(-1)^2 + 4(-1) + 1 = -3 - 4 + 1 = -6$$

and

$$f(1) = -3 \times 1^2 + 4 \times 1 + 1 = 2$$

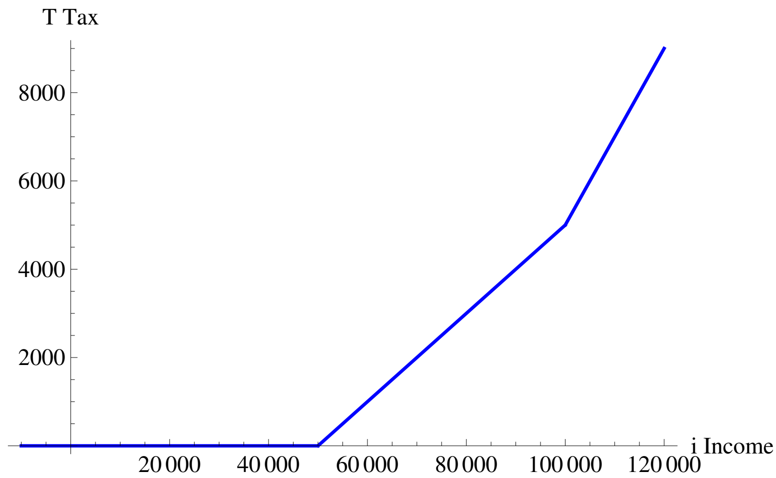
Thus $f(-1) \neq f(1)$, proving that f is not even. And $f(-1) \neq -f(1)$, proving that f is not odd.

5. You have been employed by the receiver of revenue to improve their efficiency. The first thing they want from you is a formula that gives the tax paid on income earned. Here is the information they give you.

   - No tax is paid on any income that is less than or equal to R50 000.
   - For income greater than R50 000 and less than or equal to R100 000, you pay 10% on the excess above R50 000.
   - For income greater than R100 000 you pay the tax on R100 000 as defined above plus 20% on the excess above R100 000.

   Write down a (piecewise defined) function that gives the tax paid on any income. (The above figures are not correct, but illustrate the general idea about income tax.)
   Also draw a graph of tax paid against income earned.

   Solution: Let the tax paid be denoted by T and let the income be denoted by i. Then
   $$T(i) = \begin{cases} 0 \text{ if } i \leq 50000 \\ (i - 50\,000) \times 0.1 \text{ if } 50\,000 < i \leq 100\,000 \\ 5\,000 + (i - 100\,000) \times 0.2 \text{ if } i > 100\,000 \end{cases}$$
   Here is the graph of this function:



6. (a) Solve $|-3x+4|=1$.

Solution: First remember that $-3x+4\geq 0 \Leftrightarrow x\leq 4/3$, in this case, we have $|-3x+4|=-3x+4$. Moreover, $-3x+4\leq 0 \Leftrightarrow x\geq 4/3$, in this case we have $|-3x+4|=-(-3x+4)=3x-4$. Thus,

$$\begin{array}{rcll} |-3x+4| & = & -3x+4, & x\leq 4/3 \\ & = & 3x-4 & x\geq 4/3 \end{array}$$

We have $-3x+4=1 \Leftrightarrow x=1\leq 4/3$ and $3x-4=1 \Leftrightarrow x=5/3\geq 4/3$.

(b) Solve $|-3x+4|<|x^2-2|$. (This is a hard question if you understand it it is great I will not ask you something so hard during any test unless I help you a lot with it)

Solution: First, we have

$$|-3x+4|<|x^2-2| \Leftrightarrow |-3x+4|-|x^2-2|<0$$

Note that

$$x^2-2\geq 0 \Leftrightarrow x\leq -\sqrt{2} \text{ or } x\geq \sqrt{2}$$

in this case $|x^2-2|=x^2-2$.

$$x^2-2\leq 0 \Leftrightarrow -\sqrt{2}\leq x\leq \sqrt{2}$$

in this case $|x^2-2|=-x^2+2$. Thus we have,

$$\begin{array}{rcll} |x^2-2| & = & x^2-2, & x\leq -\sqrt{2} \text{ or } x\geq \sqrt{2} \\ & = & -x^2+2, & -\sqrt{2}\leq x\leq \sqrt{2} \end{array}$$

and we have just seen that

$$\begin{array}{rcll} |-3x+4| & = & -3x+4, & x\leq 4/3 \\ & = & 3x-4 & x\geq 4/3 \end{array}$$

Now, see that $-\sqrt{2} \leq 4/3 \leq \sqrt{2}$. Thus

$$\begin{array}{ll} & |-3x+4| - |x^2-2| \\ = & -3x+4-(x^2-2) = -x^2-3x+6, \quad x \leq -\sqrt{2} \\ = & -3x+4+(x^2-2) = x^2-3x+2, \quad -\sqrt{2} \leq x \leq 4/3 \\ = & 3x-4+(x^2-2) = x^2+3x-6, \quad 4/3 \leq x \leq \sqrt{2} \\ = & 3x-4-(x^2-2) = -x^2+3x-2, \quad x \geq \sqrt{2} \end{array}$$

Help: Remember that when you have a quadratic polynomial ax^2+bx+c with discriminant $\Delta = b^2-4ac$. If $\Delta \leq 0$ then ax^2+bx+c has the same sign as a for all x. If $\Delta > 0$, then we have two roots for this polynomial

$$x_1 = \frac{-b-\sqrt{\Delta}}{2a} \textit{ and } x_2 = \frac{-b+\sqrt{\Delta}}{2a}$$

In this case, ax^2+bx+c has the same sign of a outside of the roots and the opposite sign of a in between the roots.

The discriminant of $-x^2-3x+6$ is $(-3)^2-4\times(-1)\times 6 = 9+24 = 35$, thus it has two roots $-(3-\sqrt{35})/2 = (-3+\sqrt{35})/2$ and $-(3+\sqrt{35})/2 = (-3-\sqrt{35})/2$. Thus, $-x^2-3x+2 < 0 \Leftrightarrow x < (-3-\sqrt{35})/2$ *and* $x > (-3+\sqrt{35})/2$. In particular, when $x \leq \sqrt{2}$, $|-3x+4|-|x^2-2| = -x^2+3x-2 < 0$, when $x < (-3-\sqrt{35})/2$.

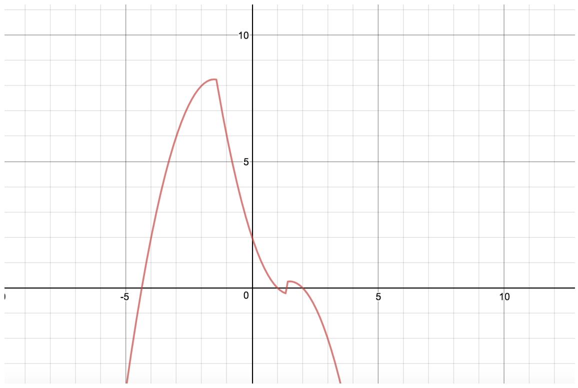
The discriminant of x^2-3x+2 is $(-3)^2-4\times 1\times 2 = 1$, thus it has two roots $(3-1)/2 = 1$ and $(3+1)/2 = 2$. Thus, $-x^2+3x-2 < 0 \Leftrightarrow 1 < x < 2$. In particular, when $-\sqrt{2} \leq x \leq 4/3$, $|-3x+4|-|x^2-2| = -x^2-3x+2 < 0$ when $1 < x \leq 4/3$.

The discriminant of x^2+3x-6 is $3^2-4\times 1\times(-6) = 9+24 = 33$, thus it has two roots $(-3-\sqrt{33})/2$ and $(-3+\sqrt{33})/2$. Thus, $-x^2+3x-2 < 0 \Leftrightarrow (-3-\sqrt{33})/2 > x$ *and* $x > (-3+\sqrt{33})/2$. In particular, when $4/3 \leq x \leq \sqrt{2}$, $-x^2+3x-2 < 0$ when $4/3 \leq x < (-3+\sqrt{33})/2$.

The discriminant of $-x^2+3x-2$ is $3^2-4\times(-1)\times(-2) = 9-8 = 1$, thus it has two roots $-(-3-1)/2 = 2$ and $-(-3+1)/2 = 1$. Thus, $-x^2+3x-2 < 0 \Leftrightarrow x < 1$ *and* $x > 2$. In particular, over $[\sqrt{2}, +\infty)$, $|-3x+4|-|x^2-2| = -x^2+3x-2 < 0$ is true when $x > 2$.

As a conclusion, $|-3x+4|-|x^2-2| < 0$, over $(-\infty, (-3-\sqrt{35})/2) \cup (1, (-3+\sqrt{33})/2) \cup (2, +\infty)$.

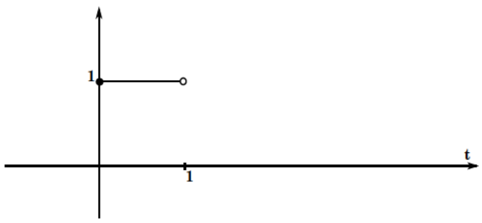
(c) Sketch the graph of $y = |-3x+4| - |x^2-2|$.



7. Let $S : \mathbb{R} \to \mathbb{R}$ defined by $S(t) = 1$ if $0 \leq t < 1$ and $S(t) = 0$ for all other values of t.

(a) Draw the graph of S: first draw a set of axes with the horizontal axis as the t-axis, then plot a few points. Finally decide what the entire graph looks like. (This is the graph that we are going to push around in this question; get the sketch of the graph checked.) What is the domain of S? What is its range?

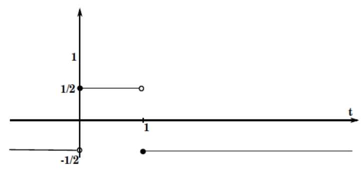
Solutions: Sketch of S(t):



By definition, the domain of S is $\mathbb{R}$.
The range of S is

$$\begin{aligned} Range(S) &= \{S(x) : x \in \mathbb{R}\} \\ &= \{S(x) : x \in [0,1]\} \cup \{S(x) : x \in \mathbb{R}\backslash[0,1]\} \\ &= \{1 : x \in [0,1]\} \cup \{0 : x \in \mathbb{R}\backslash[0,1]\} \\ &= \{0,1\} \end{aligned}$$

(b) Let $T : \mathbb{R} \to \mathbb{R}$ defined by $T(t) = S(t) - 1/2$. Draw the graph of T using the same set of axes used in the previous question. What have we really done to the graph of S to get the graph of T?
Solutions:



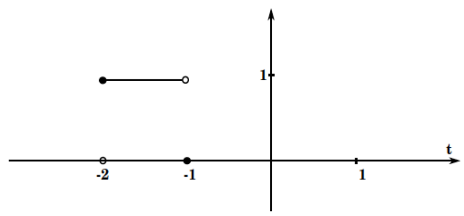
We have shifted the graph of S 1/2 downward.

(c) Let c be a constant, and $U : \mathbb{R} \to \mathbb{R}$ defined by $U(t) = S(t) + c$. Explain how the graph of U can be obtained from the graph of S. (If you're stuck, look again at the previous question; we have tried to generalise in this question what we did for a specific value of c in the previous question.)

Solutions: $Graph(U)$ is the same as $Graph(S)$ shifted up by c units if c is positive; down by $|c|$ units if c is negative.

(d) Let $W : \mathbb{R} \to \mathbb{R}$ defined by $W(t) = S(t+2)$. Draw the graph of S and W on a new set of axes. If you're stuck, use a table of values for t; try to see what we have done to S here to get W.

Solutions:



(e) Let $X : \mathbb{R} \to \mathbb{R}$ defined by $X(t) = S(t-2)$. Draw the graph of X in on the set of axes for your last question. (Why have we asked this question? What's the difference between this question and the last? Get your graphs for W and X checked.)

Solutions: $Gaph(X)$ is the same as $Graph(S)$ but shifted 2 units to the right.

(f) Let a be a constant and $Y : \mathbb{R} \to \mathbb{R}$ defined by $Y(t) = S(t + a)$. Give a rule for obtaining the graph of Y from S. (Hint: the two previous questions should help here.)

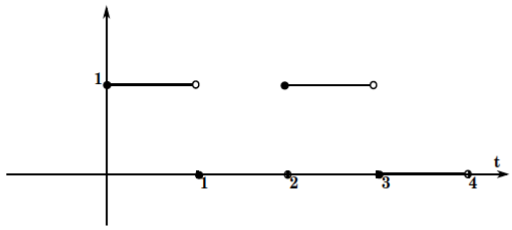
Solutions: $Graph(Y)$ is obtained from $Graph(S)$ by shifting a units to the left if a is positive and $|a|$ units to the right if a is negative.

(g) Let $Z : \mathbb{R} \to \mathbb{R}$ defined by $Z(t) = S(S(t))$ (What does that mean? If in doubt, ask!) Draw the graph of Z on a new set of axes. (Use tables if you are stuck.)

Solution: For any $t \in [0,1)$, then $Z(t) = S(S(t)) = S(1) = 0$ and for any $t \in \mathbb{R}\backslash[0,1)$, we have $Z(t) = S(S(t)) = S(0) = 1$. Thus, you can now draw the graph of Z easily.

(h) Let $A : [0,4) \to \mathbb{R}$ defined by $A(t) = S(t)$ for $0 \leq t < 2$ and let $A(t) = A(t-2)$ for $2 \leq t < 4$. ($S(t)$ is the same function used in the previous two questions.)

i. Sketch the graph of A on a new set of axes. What is the domain of A?
Solutions: The domain of A is $[0,4)$.



ii. Let $A : [0,+\infty) \to \mathbb{R}$ defined by $B(t) = S(t)$ for $0 \leq t < 2$ and $B(t) = B(t-2)$ for all $t > 2$. Draw the graph of B on a new set of axes. (Do you have enough information about B? Try to calculate $B(3)$, $B(5)$; this may give you a hint.)
Solutions: The first bit of A now repeats infinitely to the right.

iii. Suppose we let $A : \mathbb{R} \to \mathbb{R}$ defined by $C(t) = S(t)$ for $0 \leq t < 2$, and $C(t) = C(t-2)$ for all $t \notin [0,2)$. Draw a graph of C. (By the way: if you think this is an incredibly stupid function to be thinking about, don't do electrical engineering!)

Solutions: The graph of C now extends infinitely to the left as well.

8. Describe (domain, codomain, rule) $g \circ f$ and $f \circ g$ for the following pairs of functions and decide if those are equals or not:

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x^2 - 5$,
Solution: We have $g \circ f : \mathbb{R} \to \mathbb{R}$ defined by

$$g \circ f(x) = g(f(x)) = g(3x) = 2(3x)^2 - 5 = 2 \times 9x^2 - 5 = 18x^2 - 5$$

and $f \circ g : \mathbb{R} \to \mathbb{R}$ defined by

$$f \circ g(x) = f(g(x)) = f(2x^2 - 5) = 3(2x^2 - 5) = 6x^2 - 15$$

Note that $g \circ f(0) = -5$ and $f \circ g(0) = -15$, thus $g \circ f(0) \neq f \circ g(0)$. That proves that $g \circ f \neq f \circ g$.

(b) $f : \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^{4x}$ and $g : \mathbb{R}^+ \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$,

Solution: We have $g \circ f : \mathbb{R} \to \mathbb{R}$ defined by

$$g \circ f(x) = g(f(x)) = g(e^{4x}) = \sqrt{e^{4x}} = e^{2x}$$

and $f \circ g : \mathbb{R} \to \mathbb{R}$ defined by

$$f \circ g(x) = f(g(x)) = f(\sqrt{x}) = e^{4\sqrt{x}}$$

Note that $g \circ f(1) = e^2$ and $f \circ g(1) = e^4$, thus $g \circ f(1) \neq f \circ g(1)$. That proves that $g \circ f \neq f \circ g$.

9. Decompose the following functions into the form $g \circ f$ (describe fully $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, there is not only one answer possible so you might be right and still have a different result from your friend):

   (a) $6x + 3$,

   Solution : You could take $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 6x$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x + 3$.

   (b) $4x^2$

   Solution : You could take $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 4x$.

   (c) $4x^2$ (in a different way),

   Solution : You could take $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.

   (d) $e^x + 4$,

   Solution : You could take $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x + 4$.

   (e) $x^2 + 2x + 1$.

   Solution : You could take $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.

10. Let

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \to & -3x + 10 \end{array},$$

   (a) Prove that f is onto.

   Solution: Let $y \in \mathbb{R}$. We looking for $x \in \mathbb{R}$ such that $f(x) = y$. If such a x exists then we have $-3x + 10 = y$. That means that $x = \frac{y-10}{-3}$. Now take $x = \frac{y-10}{-3} \in \mathbb{R}$, since $y \in \mathbb{R}$, then

$$f(x) = f(\frac{y - 10}{-3}) = -3\frac{y - 10}{-3} + 10 = y$$

   We have proven that any element in the codomain has a preimage in the domain thus f is onto.

(b) Prove that f is one-to-one.

Solution: Let $x_1, x_2 \in \mathbb{R}$, $f(x_1) = f(x_2)$ implies $-3x_1 + 10 = -3x_2 + 10$ thus $-3x_1 = -3x_2$ and $x_1 = x_2$. Proving that f is one-to-one.

(c) Is f bijective?

Solution : Since f is one-to-one and onto thus f is bijective.

(d) Compute the inverse of f.

Solution : We have seen in class that the inverse map goes from the codomain of f to the domain of f sending a element of the codomain of f to its unique preimage. Thus the inverse of f is the map $f : \mathbb{R} \to \mathbb{R}$ with the rule $f(x) = \frac{x-10}{-3}$.

(e) Verify that the inverse computed in the previous question is indeed an inverse using the identity seen in class for inverses with composition.

Solution : We have seen in class that the inverse of f f^{-1} is characterized by the property

$$f \circ f^{-1} = Id_{\mathbb{R}} \text{ } and \text{ } f^{-1} \circ f = Id_{\mathbb{R}}$$

We have

$$f \circ f^{-1}(x) = f(f^{-1}(x)) = f(\frac{x-10}{-3}) = -3\frac{x-10}{-3} + 10 = x$$

and

$$f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(-3x+10) = \frac{-3x+10-10}{-3} = x$$

Thus, we prove the identity and confirmed that f^{-1} is the inverse of f.

11. Let

$$f : \begin{array}{ccc} [0,+\infty) & \to & [0,+\infty) \\ x & \to & x^2 \end{array},$$

(a) Prove that f is onto.

Solution: Let $y \in [0,+\infty)$. We looking for $x \in [0,+\infty)$ such that $f(x) = y$. If such a x exists then we have $x^2 = y$. Remember, since $y > 0$, we have $y = x^2 \Leftrightarrow x = \sqrt{y} \text{ } or x = -\sqrt{y}$. But only $\sqrt{y} \in [0,+\infty)$ Now take $x = \sqrt{y} \in [0,+\infty)$, then

$$f(x) = f(\sqrt{y}) = \sqrt{y}^2 = y$$

We have proven that any element in the codomain has a preimage in the domain thus f is onto.

(b) Prove that f is one-to-one.

Solution: Let $x_1, x_2 \in [0, +\infty)$, $f(x_1) = f(x_2)$ implies $x_1^2 = x_2^2$ thus $x_1^2 - x_2^2 = 0$ and $(x_1 - x_2)(x_1 + x_2) = 0$. Thus either $x_1 = x_2$ or $x_1 = -x_2$. Since x_1 and x_2 are both positive, the only possibility is that $x_1 = x_2$. Proving that f is one-to-one.

(c) Is f bijective?

Solution : Since f is one-to-one and onto thus f is bijective.

(d) Compute the inverse of f.

Solution : We have seen in class that the inverse map goes from the codomain of f to the domain of f sending a element of the codomain of f to its unique preimage. Thus the inverse of f is the map $f : \mathbb{R} \to \mathbb{R}$ with the rule $f(x) = \sqrt{x}$.

(e) Verify that the inverse computed in the previous question is indeed an inverse using the identity seen in class for inverses with composition.

Solution : We have seen in class that the inverse of f f^{-1} is characterized by the property

$$f \circ f^{-1} = Id_{\mathbb{R}} \text{ and } f^{-1} \circ f = Id_{\mathbb{R}}$$

We have

$$f \circ f^{-1}(x) = f(f^{-1}(x)) = f(\sqrt{x}) = \sqrt{x}^2 = x$$

and

$$f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = x$$

Thus, we prove the identity and confirmed that f^{-1} is the inverse of f.

12. **Here you find hint solutions, I leave you to complete the justifications**

A certain bacteria colony is know to have a doubling time of 3 hours. Suppose that you are infected with 50 of these bacteria when you cut yourself on something. What is the size of the population after 15 hours? After t hours? Give the size of the population after 20 hours.

Hint solution: Let $B(t)$ stand for the number of bacteria at time t (in hours).Then

$$B(t) = 50 \times 2^{t/3}$$

is the formula for $B(t)$ at any time t. Use this to calculate $B(20)$.

13. **Here you find hint solutions, I leave you to complete the justifications**

You sell hot chocolate at the waterfront, and you find that sales vary with the time of year. The lowest sales, of 50 litres a day, happen on 1 February. The highest sales, of 350 litres a day, happen on 1 August. We want to model this situation with a sine curve, where $t = 0$ is the start of January. You will see that this is actually an exercise in shifting and scaling the standard sine function.

(a) Begin by drawing a graph that covers the period of a year, putting in the in- formation given above. Units for the horizontal axis should be months. (12 in a year?) We would now like a formula for the amount of hot chocolate sold at any time, using this sine curve model. Use the function

$$f(t) = [asin(bt - c)] + d$$

where t is time measured in months, and $f(t)$ is the number of litres of hot chocolate sold at time t. You need to find out what the correct constants a, b, c, d are, for this situation.

Hint solution: Your peak occurs at the value 7 (end of month 7) and the trough (least value) occurs at value 1 (end of month 1).

Find them by following these steps:

(b) What is the amplitude you want for your graph? Since the ordinary sine function has amplitude 1, you will need to stretch the sine function vertically to get the amplitude you want. How much do you need to stretch? Which constant have you just worked out?

Hint solution: The required amplitude is $3 \times 50 = 150$. So we have stretched vertically by 150. This is the value of a.

(c) If you take an ordinary (stretched) sine function, it oscillates about the x axis. You want your function to oscillate around what value? That means that you want to shift your function. This tells you another constant. Which one?

Hint solution: The sales oscillate about a mean value of 200; shift up by 200. This is d.

(d) That leaves b and c to be found. Do that as follows: substitute in the co-ordinates of the points that correspond to 1 February and 1 August. That leaves you with two equations in two unknowns. Solve them simultaneously. (You will get many solutions; just pick one for b and one for (c).

Hint Solution Substitute in $(1, 50)$ and $(7, 350)$. You get $b = 30$ and $c = 120$. (Alternatively, you might get $b = ?30$ and $c = 60$, but that is still the same function: why?)

(e) Write out the final formula for $f(t)$.

Hint solutions: Replace the values of a, b, c, d and you will find $f(t)$.

14. Find $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ for f and g with rule $f(x) = \sqrt{x}$, $g(x) = \sqrt{1-x}$. What is the domain of definition of f and g (the maximal interval of $\mathbb{R}$ such that f and g are defined)? Give the domains over which the composite function above are defined. Describe some domain and codomain you could chose so that you can compose them. (several answers are possible) Compute those composition.

Solution: The domain of definition of f is $[0, +\infty)$ as f is a square root and square roots of negative real number do not exist and the domain of definition of g is the set of $x \in \mathbb{R}$ such that $1 - x \geq 0$, for the same reason as before, that is $x \leq 0$.

In order to be able to do $f \circ g$ we need the codomain g to be equal to the domain of f. But we have see that the domain of f is $[0, +\infty)$, and also $g(x) = \sqrt{1-x} \in [0, +\infty)$. So we could take $g : (-\infty, 0] \to [0, +\infty)$ and $f : [0, +\infty) \to \mathbb{R}$, and then the codomain of g is equal to the domain of f which permits to be able to do the composition. Moreover,

$$f \circ g(x) = f(g(x)) = f(\sqrt{1-x}) = \sqrt{\sqrt{1-x}}$$

In order to be able to compute $g \circ f$ we need the codomain of f to be equal to the domain of g. But we have seen that g is only defined over $(-\infty, 1]$, so we can only consider subinterval as this interval for the domain of g. Also, the out puts of f belong to $[0, +\infty)$, in order to choose a domain for g such that the domain of g is the codomain of f a possibility maybe would be to take the intersection $(-\infty, 1] \cap [0, +\infty) = [0, 1]$ as the domain of f and codomain of g. But now we need to think more because we need to find a domain of g such that the outputs $g(x)$ belong to $[0, 1]$. So the question is where vary x when $g(x) \in [0, 1]$.

$$\begin{aligned} 0 \leq g(x) \leq 1 \quad &\Leftrightarrow \quad 0 \leq \sqrt{1-x} \leq 1 \\ &\Leftrightarrow \quad 0 \leq 1 - x \leq 1 \\ &\Leftrightarrow \quad -1 \leq -x \leq 0 \\ &\Leftrightarrow \quad 0 \leq x \leq 1 \end{aligned}$$

In conclusion, if we want to chose the codomain of g to be $[0, 1]$ we can only pick its domain inside $[0, 1]$. So we can take $g : [0, 1] \to [0, 1]$ and $f : [0, 1] \to \mathbb{R}$, Now everything makes sense and I can compose and

$$g \circ f(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{1 - \sqrt{x}}$$

In order to be able to compute $f \circ f$, we need to choose the domain of f equal to the codomain of f. We have seen that the domain of f can be chosen inside $[0,+\infty)$ also the codomain of f belongs to $[0,+\infty)$ thus let's chose the domain of f equal to the codomain of f equal to $[0,+\infty)$. Thus $f:[0,+\infty)\to[0,+\infty)$ and

$$f\circ f(x)=f(f(x))=f(\sqrt{x})=\sqrt{\sqrt{x}}.$$

Finally, in order to compute $g\circ g$, we should to choose the domain of g equal to codomain of g, but the domain of g is included in $(-\infty,1]$ and the values $g(x)$ belong then to $[0,+\infty)$. As before we could choose the intersection $[0,1]$ as codomain and domain of g and we are able to compose

$$g\circ g(x)=g(g(x))=\sqrt{1-\sqrt{1-x}}$$

15. Express the following functions in the form $f\circ g$ (that is, give an f and g that will work):

    (a) $h:\mathbb{R}\to\mathbb{R}$ defined by $h(t)=\sqrt{1+t^2}$,
    Solution: You could take $g:\mathbb{R}\to\mathbb{R}$ defined by $g(t)=t^2$ and $f:\mathbb{R}\to\mathbb{R}$ defined by $f(t)=1+t$. Then, for any $t\in\mathbb{R}$,

    $$f\circ g(t)=f(g(t))=f(t^2)=1+t^2=h(t)$$

    (b) $h:\mathbb{R}\to\mathbb{R}$ defined by $h(t)=cos^3(t)$,
    Solution: You could take $g:\mathbb{R}\to\mathbb{R}$ defined by $g(t)=cos(t)$ and $f:\mathbb{R}\to\mathbb{R}$ defined by $f(t)=t^3$. Then, for any $t\in\mathbb{R}$,

    $$f\circ g(t)=f(g(t))=f(cos(t))=cos^3(t)=h(t)$$

    (c) $h:\mathbb{R}\to\mathbb{R}$ defined by $h(t)=\frac{cos(t)}{cos^2(t)+2}$.
    Solution: You could take $g:\mathbb{R}\to\mathbb{R}$ defined by $g(t)=cos(t)$ and $f:\mathbb{R}\to\mathbb{R}$ defined by $f(t)=\frac{t}{t^2+2}$. Then, for any $t\in\mathbb{R}$,

    $$f\circ g(t)=f(g(t))=f(cos(t))=\frac{cos(t)}{cos^2(t)+2}=h(t)$$

16. Given that $g:\mathbb{R}\to\mathbb{R}$ defined by $g(x)=2x+1$ and $h:\mathbb{R}\to\mathbb{R}$ defined $h(x)=4x^2+4x+7$, find a function f such that $f\circ g=h$.

    Solution: First observe that $(2x+1)^2=4x^2+4x+1$ and thus $h(x)=(2x+1)^2+6$. Take $f:\mathbb{R}\to\mathbb{R}$ with $f(x)=x^2+6$. Then you can check that for any $x\in\mathbb{R}$,

    $$f\circ g(x)=f(g(x))=f(2x+1)=(2x+1)^2+6=h(x)$$